Nonlinear electron motion in a coherent whistler wave packet

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Map equations are derived, with which nonlinear electron motion in a coherent whistler wave packet is investigated. All solutions of these equations belong to a certain strange attractor and describe chaotic motion with the stable means. The class of solutions determined the intermittent dynamics as the control parameter of the wave-particle system increases above the appropriate critical value is found. An application of the results to the problem of the stability of Earth's radiation belts is considered. It is shown that the efficient acceleration processes take place for relativistic electrons of a few MeV. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2959121]

I. INTRODUCTION

The goal of the present work is to describe high-energy electron motion in a coherent packet of whistler modes. Stochastic dynamics of charged particles in the field of a wave packet is one of the fundamental problems in the theory of plasma physics.[1] Chaotic dynamics of relativistic particles in the spectrum of waves is of particular interest. Chernikov *et al.*[2] have looked at dynamics chaos as a fundamental property of the wave-particle system, describing chaos in the relativistic generalization of the standard map, based on the problem of particle acceleration in the electrostatic field of a time-like wave packet. Klimov and Tel'nikhin[3] and Krotov and Tel'nikhin[4] have shown that stochastic heating of relativistic particles by Langmuir waves in space plasmas can be regarded as a possible mechanism for the formation of the energy spectrum of cosmic rays. They have also studied the evolution of the distribution function caused by the stochasticity. Nagornykh and Tel'nikhin[5] have developed the relativistic theory for the stochastic motion of electrons in the presence of obliquely propagating electrostatic waves. For that case, the ambient magnetic field plays an important role in randomizing the phase of the particles with respect to the wave phase. The conditions under which a magnetized ion can be accelerated through a nonlinear interaction with a pair of beating electrostatic waves have been explored by Benisti *et al.*,[6] who also have shown that nonlinear ion acceleration in that physical situation is always a stochastic process. On the other hand, stochastic particle heating by a spectrum of electromagnetic waves propagating transversely to the magnetic field has been extensively studied.[7]

Stochastic motion of relativistic electrons in the whistler wave packet with application of the results to electron heating in the Jovial magnetosphere was studied in Ref. 8.

This paper investigates electron motion in a wave packet as a function of the magnitude of the wave field. We develop the theory proposed by Khazanov *et al.* in Ref. 9 to describe the electron motion in a relatively strong wave field, employing a Hamiltonian formalism in which desired solutions are the result of an appropriate canonical transformation on corresponding manifolds. As shall be shown, several different local coordinates systems are found that describe complementary aspects of the phase dynamics.

The paper is organized as follows. In Sec. II, the canonical equation of motion in terms of the action-angle variables is derived. In Sec. III, the dynamics are represented as a successive action of the one-parameter group of transformation acting on a strange attractor (SA), the canonical status of the variables on SA is demonstrated, and the dynamical and structural invariants such as the Kolmogorov entropy and fractal measure are calculated, showing that the topological equivalence criterion defines well the extreme value of the energy spectrum. In Sec. IV, a class of solutions describing the so-called intermittent dynamics is described, and the solution shows that the sudden appearance of regular orbits is conditioned by bifurcation of the vector field as the control parameter increases above the appropriate critical value. Also, due to nonadiabatic behavior of the orbits near the boundaries of the SA, the regular motion is accompanied by orbital drift in phase space. Section V summarizes the results.

II. BASIC EQUATIONS

Let us consider a relativistic particle of charge $|e|$ and mass m in the wave packet of extraordinary electromagnetic waves propagating along an external uniform magnetic field of strength B. The Hamiltonian corresponding to the problem is

$$H(\mathbf{r},\mathbf{p};t) = \sqrt{m^2 + (\mathbf{p}+\mathbf{A})^2} \tag{1}$$

and the canonical equations of motion are

$$\dot{\mathbf{p}} = [\mathbf{p},H], \quad \dot{\mathbf{r}} = [\mathbf{r},H], \tag{2}$$

where $\mathbf{p}$ is the particle momentum, $\mathbf{r}$ is the position vector, and $\mathbf{A}=\mathbf{A}^w+\mathbf{A}^{\text{ext}}$ is the vector potential, with the superscripts w and ext denoting the wave and external fields, respectively; and [,] stands for the Poisson brackets. We have employed here and throughout the units in which the speed of

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light $c=1$ and charge $|e|=1$. We denote by R the set of all real numbers. Then, $\mathbf{p}\in R^3$, $\mathbf{r}\in R^3$, and the smooth manifold $M=R^6$ will be a canonical space of this dynamic system, and $R^6=R^3\times R^3$ is a direct product space.

In order to write down equations of the particle motion, one must specify a coordinate system. We have chosen a Cartesian spatial coordinates system whose z axis is directed along the external magnetic field. Making use of the Maxwell relations

$$\mathbf{B}=\mathrm{rot}\mathbf{A}, \quad \mathbf{E}=-\partial\mathbf{A}/\partial t, \tag{3}$$

we have in the coordinate representation

$$\mathbf{r}=(x,y,z), \quad \mathbf{B}^{\mathrm{ext}}=(0,0,B), \tag{4}$$

$$\mathbf{A}^{w}=\left(\sum_k A_k \sin\varphi, \sum_k A_k \cos\varphi, 0\right), \qquad \varphi = zk - t\omega_k, \tag{5}$$

$$\mathbf{A}^{\mathrm{ext}}=(-By,Bx,0)/2. \tag{6}$$

Here the expression for $\mathbf{A}^{\mathrm{ext}}$ is written in the axial gauge, A_k is the amplitude of mode in the wave packet, k is the wave number, and ω_k is the dispersion equation.

The dispersion relation for the electron branch of the whistler mode waves in the cold magnetoplasma is written as

$$k^2/\omega^2 = 1+\omega_p^2/[\omega(\omega_B-\omega)], \tag{7}$$

which in the long-wavelength approximation $(\omega_B\omega/\omega_p^2)\ll 1$ reduces to

$$v_{\mathrm{ph}}^2=\omega(\omega_B-\omega)/\omega_p^2, \quad v_{\mathrm{ph}}^2\ll 1, \tag{8}$$

where ω_B and ω_p are the gyrofrequency and electron plasma frequency, respectively. On account of axial symmetry of the nonperturbative problem, we introduce the new variables, an action (I), and an angle (θ), by a canonical transformation $(x,p_x;y,p_y)\rightarrow(\theta,I)$,

$$x=r\cos\theta, \quad p_x=-(mr\omega_B/2)\sin\theta, \qquad y=r\sin\theta, \quad p_y=(mr\omega_B/2)\cos\theta; \tag{9}$$

$$r=\sqrt{2m\omega_B I}/m\omega_B, \quad \omega_B=B/m, \tag{10}$$

where r is the gyroradius.

The Hamiltonian (1) in this representation becomes

$$H(z,p_z;\theta,I;t)=H_0(p,I)+\sqrt{2m\omega_B I}H_0^{-1}\cdot\sum_k A_k\cos(zk+\theta-t\omega_k), \tag{11}$$

$$H_0(p,I)=\sqrt{m^2+p^2+2m\omega_B I}. \tag{12}$$

Here, we have assumed that the ratio $\mu=A/m(\ll 1)$ is the small parameter of the problem, and retain in Eq. (11) only the leading terms.

Associated with Eq. (11), the equations of motion are

$$\dot{p}=[p,H]=\sqrt{2m\omega_B I}H_0^{-1}\cdot\sum_k kA_k\sin\psi_k, \tag{13}$$

$$\dot{I}=[I,H]=\sqrt{2m\omega_B I}H_0^{-1}\cdot\sum_k A_k\sin\psi_k, \tag{14}$$

$$\dot{z}=[z,H]=pH_0^{-1}, \quad \dot{\theta}=[\theta,H]=\omega_B m H_0^{-1}. \tag{15}$$

In Eq. (15), we omit the terms of the order of μ^2 and introduce the definition for the phase

$$\psi_k \overset{\mathrm{def}}{=} zk+\theta-\omega_k t. \tag{16}$$

We will discuss a relativistic electron motion, therefore the expression for the wave field in Eq. (11) can be given in the so-called space-like representation,[8,10]

$$A^w(t,z)=A\exp(i\psi)\sum_{n\in Z}\delta(\zeta-n), \qquad \psi=zk+\theta-t\omega. \tag{17}$$

Here, the Poisson sum formula

$$\sum_{n\in Z}\exp(in\Delta kz)=\sum_{n\in Z}\delta(\zeta-n) \tag{18}$$

has been employed. A,ω, and k are the magnitude, frequency, and wave number of the fundamental (characteristic) mode, $\zeta=(z/L)(\mathrm{mod}\ 1)$, L is the characteristic space scale, $\delta_n\equiv\delta(\zeta-n)$, $\delta(\cdot)$ is the Dirac delta function, and Z denotes the set of all integers. Note that such a wave packet manifests itself as a periodic sequence of impulses with characteristic spatial period $L=2\pi/\Delta k$, where $\Delta k/k=2\pi/N$, $(N\gg 1)$, and N is a characteristic number of modes in the wave packet.

In this approach, we write down the equations of motion (13)–(15) in the form

$$\dot{p}=kA\sqrt{2m\omega_B I}H_0^{-1}\sin\psi\sum_{n\in Z}\delta(\zeta-n), \tag{19}$$

$$g^t:\ \dot{I}=\sqrt{2m\omega_B I}H_0^{-1}A\sin\psi\sum_{n\in Z}\delta(\zeta-n), \tag{20}$$

$$\dot{z}=pH_0^{-1}, \quad \dot{\theta}=\omega_B m H_0^{-1}, \tag{21}$$

$$H_0(p,I)=\sqrt{m^2+p^2+2m\omega_B I}, \tag{22}$$

$$\dot{\psi}=\omega(p,I)=kpH_0^{-1}+\omega_B mH_0^{-1}-\omega. \tag{23}$$

In what follows, we are interested in the behavior of system (19)–(23) under the resonance conditions

$$\omega(p,I)=kpH_0^{-1}+\omega_B mH_0^{-1}-\omega=0. \tag{24}$$

First, we observe that the phase flow on M^4 $(z,p;\theta,I)\in M^4$, given by Eqs. (19)–(21), is invariable under the translation of a phase point with respect to z (mod L), and possesses the integral invariant of motion

$$p-kI=\mathrm{inv}, \tag{25}$$

reducing the phase space dimension to 2. Let the variable z and p be the canonical pair on the reduced space of orbits, $M_I^2=(U\times S)_I$, $p\in U\subset R$, z mod $L\in S$, where all orbits are parametrized by the values of I, given by the relation

$$2\omega_B I = \alpha p, \quad \alpha = (2\omega_B/\omega)v_{\rm ph}. \tag{26}$$

We set the constant of integration equal to zero, because the phase flow, along with Eq. (26) being an analytical invariant, is invariable under the transformation $H \to H + \text{const}$.

TM and T^* denote the tangent and cotangent fiberings, respectively. According to Eqs. (19)–(21), there exist the mappings[11]

$$T^M \to M: \ \bar{v} = -\dot{p}\partial/\partial p + \dot{z}\partial/\partial z, \tag{27}$$

$$T^*M \to M: \ \tilde{v} = -\dot{p}\tilde{d}z + \dot{z}\tilde{d}p, \tag{28}$$

which define the field of vectors, $\bar{v}$, and the dual field of 1-forms with the components $\dot{p}$, $\dot{z}$ given by Eqs. (19) and (21). The notations $\partial/\partial x^i$ and $\tilde{d}x^i$ are employed here for the orthonormal coordinate basis of vectors and for the dual basis of 1-forms, such that

$$\tilde{d}x^i \frac{\partial}{\partial x^i} = \delta_i^{i'} = \begin{cases} 1, & i = i' \\ 0, & i \neq i' \end{cases},$$

where $(x^i) = (p, z)$. It is clear that given fields possess the translation symmetry, viz.,

$$\vec{V}(\psi(t), \zeta, p) = \vec{V}(\psi(t) + 2\pi, \zeta + 1, p), \tag{29}$$

which permits the representation of the dynamics as an iteration process. Indeed, it is well known that the section of the field of 1-forms, $\tilde{\partial}\tilde{v}(\bar{v}) = 0$, on a submanifold in M is equivalent to the solution of Eqs. (20) and (21).[16] Thus, this equation rewritten in the explicit form is

$$pH_0^{-1}\tilde{d}p - kA\sqrt{2m\omega_B I}H_0^{-1} \sin \psi \sum \delta(\zeta - n)\tilde{d}z = 0 \tag{30}$$

and the equation for phase gradient along the vector field is

$$\tilde{d}\psi - \omega(p)L(H_0/p)\tilde{d}\zeta = 0, \tag{31}$$

which well describe the solution trajectories in the phase space of the system.

Making use of the invariant of motion (26) and the resonance condition (24), we integrate one by one, starting from Eq. (30), resulting in equations to obtain the closed set of nonlinear difference equations,

$$\begin{aligned} v_{n+1} &= v_n + \alpha^{3/2} Nb \sin \psi_n, \\ \psi_{n+1} &= \psi_n + N(1 + 1/2 \cdot \alpha |v_{n+1}^{-2/3}| \\ &\quad - |v_{n+1}^{-2/3}| v_{\rm ph}\sqrt{1 + v_{n+1}^{4/3} + \alpha v_{n+1}^{2/3}}) \operatorname{sgn} v_{n+1} (\operatorname{mod} 2\pi), \end{aligned} \tag{32}$$

expressed in terms of the variables v, ψ, where v_{n+1} and v_n are, respectively, the values of the normed momentum at times $(n+1)$ and n, and $\psi_{n+1} - \psi_n$ is the phase shift acquired by the particle. Then, integrating Eq. (30) gives the time step, $T(n)$,

$$t_{n+1} - t_n = T(n), \quad T(n) = LE_{n+1}/|p_{n+1}|, \tag{33}$$

which is a function of n.

We have employed here the following notations:

$$v = |p/m|^{3/2} \operatorname{sgn} p/m, \quad N = [kL]. \tag{34}$$

Here N is the characteristic number of modes in the SL packet, and in writing Eq. (32) we have used the a relationship between the fields A^w and B^w,

$$A/m = \alpha b/2, \quad b = B^w/B, \tag{35}$$

which follows from Eqs. (3).

The quantity A/m has a clear physical meaning: A/m is the dimensionless representation of the ratio of the work of the wave field at one wavelength to the particle rest energy. In the relativistic limit, when the inequality $\varepsilon^2 \gg \max\{1, \alpha^2\}$ is valid, the ψ equation from Eq. (32) is simplified and this set of equations reduces to the map g^n,

$$\begin{aligned} & u_{n+1} = u_n + Q \sin \psi_n, \\ g^n: \ & \psi_{n+1} = \psi_n + (3\pi^{5/3}/2Q)|u_{n+1}|^{-2/3} \operatorname{sgn} u_{n+1} (\operatorname{mod} 2\pi), \end{aligned} \tag{36}$$

written in the variables (ψ, u), where

$$u = \pi v/v_b, \quad v_b = \left(\frac{1 - v_{\rm ph}}{3}\alpha^{5/2}N^2 b\right)^{3/5} \tag{37}$$

and the control parameter Q, involving the group velocity dispersion effect, is given by

$$Q = \pi\left(\frac{3}{1 - v_{\rm vh}}\right)^{3/5} \cdot \left(\frac{b^2}{N}\right)^{1/5}. \tag{38}$$

It should be observed that the above equations are similar to those obtained by the authors in Ref. 8. However, unlike the current approach, the manner in which the map has been derived in Ref. 9 is the technique of 1-form on extended phase space.

From Eq. (36) it follows that the dynamical system (M, g^n) is invariable under the transformation

$$\psi \to -\psi, \quad u \to -u \tag{39}$$

and the inversion of a point with respect to a circle $\psi(\text{mod}2\pi) \in S$.

III. STRANGE ATTRACTOR OF THE SYSTEM

The family of maps, $\{g^n, n \in Z\}$, where g^n is given by Eq. (36) at $\forall n \in Z$ to be the one-parametric group of transformations of M. To show this, it proves convenient to define the map on one iteration, $g_0^1(\psi_0, u_0) \equiv g^1$, where (ψ_0, u_0) is the initial state of this system. One considers the map of the extended phase space in M: $Z \times M \to M$, $n \in Z$, $(\psi, u) \in M$. Then, from Eq. (36) it follows that $g^{n+1} = g^1 g^n$, therefore by induction we conclude that the transformation g^n generates the group $g^n = (g^1)^n$. The given group acts on M whose local topology is determined by the eigenvalues of the Jacobi matrix,

$$J = \frac{\partial(u_{n+1}, \psi_{n+1})}{\partial(u_n, \psi_n)}. \tag{40}$$

It is known that the trace of the matrix $|{\rm tr}J| = 3$ corresponds to the topological modification of a phase space,[12] which, in our case, well defines the upper bound of $\{u\}$. So far as

$$\mathrm{tr}J = 2 + (\pi^5 u^{-5})^{1/3}, \tag{41}$$

therefore

$$\sup\{u\} = u_b, \quad |u_b| = \pi. \tag{42}$$

As we discussed above, we are considering the dynamics as a successive action of g^n on M. It is important to note that the Jacobian of Eq. (40) is equal to 1; therefore, g^n has a structure of the differentiable area-preserving map, expressed in terms of variables u and ψ, being the canonical pair. Moreover, we will show that the map g^n inherits the canonical structure of the original Hamiltonian system. Indeed, let (ψ,u) be the local coordinates on M, and F and G be any smooth functions of (ψ,u), which obey the commutation rule defined by the Poisson bracket,

$$[F,G] = \frac{\partial F}{\partial \psi}\frac{\partial G}{\partial u} - \frac{\partial G}{\partial u}\frac{\partial F}{\partial \psi}. \tag{43}$$

In particular, the special Poisson brackets are

$$[u,u] = [\psi,\psi] = 0, \quad [\psi,u] = 1. \tag{44}$$

We define by

$$h(\psi,u,\zeta) = Q\cos\psi \sum_n \delta(\zeta - n) + Q(9\pi^2/2Q^2) \times (|u|/\pi)^{1/3}, \quad n \in Z, \tag{45}$$

the function on M. Then, due to Eqs. (43) and (45) we write down the set of dynamical equations,

$$\begin{aligned} \dot{u} &= [u,h] = Q\sin\psi \sum_n \delta(\zeta - n), \quad \dot{u} = \partial u/\partial\zeta, \\ \dot{\psi} &= [\psi,h] = (3\pi^{5/3}/2Q)|u|^{-2/3}\,\mathrm{sgn}\,u, \quad \dot{\psi} = \partial\psi/\partial\zeta, \end{aligned} \tag{46}$$

with the variable ζ playing the role of a temporal coordinate.

Integrating the set (46), we obtain the map g^n coincident with Eq. (36). So, we conclude $h(\psi,u,\zeta)$ is the Hamiltonian function on M, which is tantamount to the system, therefore the vector field, $\bar{v}(\psi,u,\zeta)$, where

$$\bar{v}(\psi,u,\zeta) = \dot{u}\,\partial/\partial u + \dot{\psi}\,\partial/\partial\psi \tag{47}$$

with the components given by Eq. (46), is equivalent the vector field given by Eq. (27).

The dynamical system $(M;g^n)$ is similar to that studied by Khazanov *et al.* in Ref. 8 In this work, it was shown that all trajectories for a broad range of control parameter Q belong to the strange attractor (SA). Shown in Fig. 1 is the strange attractor of the system. This SA is characterized by the two invariants, namely by the Kolmogorov entropy, being the kneading invariant, and the fractal measure. As far as the fractal dimension being equal to that of the phase space, the points curve evenly filling all obtainable phase space. Hence, the strange attractor is the invariant set, $g^n\mathrm{SA}=\mathrm{SA}$, tightly embedded in phase space at $n \to \infty$.

So far as the variables (ψ,u) are the canonical pair, the distribution function (probability density) $f(u;t)$ obeys the Fokker–Planck–Kolmogorov (FPK) equation

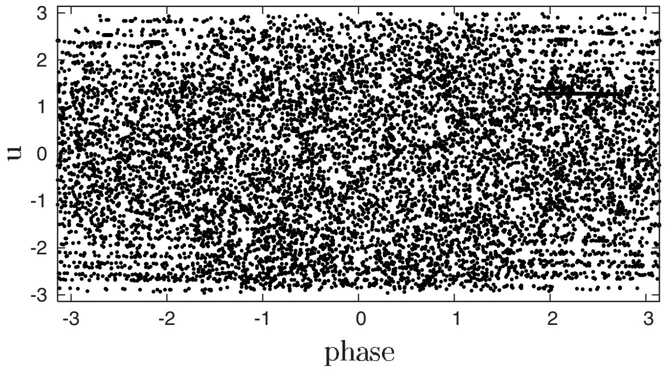

FIG. 1. The strange attractor of the system. One single trajectory of length 10^6 is obtained by numerical integration of g^n. The trajectory was started from the point $u_0=10^{-3}$, $\psi_0=10^{-4}$. The parameter Q is 0.02π.

$$\frac{\partial f(u;t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial u}D\frac{\partial f}{\partial u}. \tag{48}$$

Here, D is the conventional diffusion coefficient in phase space,

$$D = \langle (u_{n+1} - u_n)^2 \rangle T^{-1}, \tag{49}$$

in which $(u_{n+1}-u_n)$ is substituted from Eq. (36), $<\cdot>$ denotes the phase average, and T is the time scale of mapping (36). The function $f(u,t)$ is normalized by

$$\int_{-\pi}^{\pi} f(u,t)du = 1. \tag{50}$$

First, by means of Eqs. (36) and (49), we calculate the diffusion coefficient

$$D = Q^2/2T. \tag{51}$$

With the help of the result (42) from Eq. (48) we evaluate the characteristic time for redistribution u over the spectrum,

$$t_d \simeq 2u_b^2/D = T(2\pi/Q)^2. \tag{52}$$

The solution of the FPK equation in the limiting case $t \gg t_d$ with $f(u)$ and its derivative $\partial f/\partial u$ vanishing at the boundary may be given in the form of the uniform distribution,

$$f(u) = (2\pi)^{-1}. \tag{53}$$

Now we apply our results to the problem of particle acceleration in Earth's radiation belts. In accordance with the paper of Roth *et al.*,[13] in which they discussed in detail the problem of the interaction of a whistler wave with relativistic electrons, we choose a set of standard values: Scale length of background magnetic field $L=10^9$ cm, equatorial magnetic field of 10^{-3} G, the electron cyclotron frequency $\omega_B=2.6\times10^4\ \mathrm{s}^{-1}$, ratio of wave frequency to the equatorial gyrofrequency is 0.5, and $\omega_p/\omega_B=2-3$, and typical whistler amplitudes are in the range (10–100) pT, but occasionally wave amplitudes approach ln T.[14] The wave magnitudes of the outer zone chorus are small enough, resulting in weak diffusion scattering. Consequently, the electron heating should occur gradually over many drift orbits. By Eq. (8) we find the phase velocity, $v_{\mathrm{ph}}=0.25$, and the group velocity, $v_{gr}=2\cdot v_{\mathrm{ph}}(\omega_B-\omega/\omega_B)$, $v_{gr}\simeq0.25$. First we evaluate

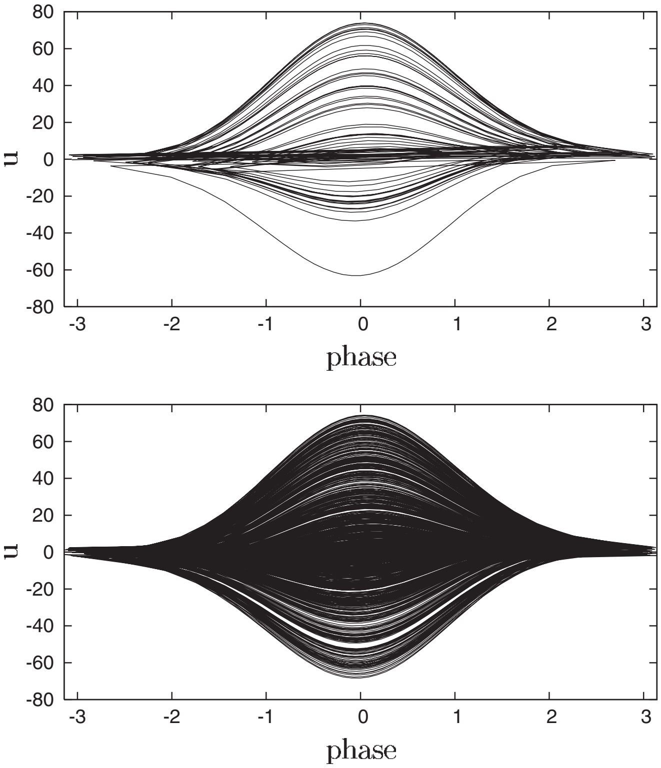

FIG. 2. Space of orbits at times (a) 10^3; (b) 10^4, respectively. $Q=3\pi/\sqrt{2}$.

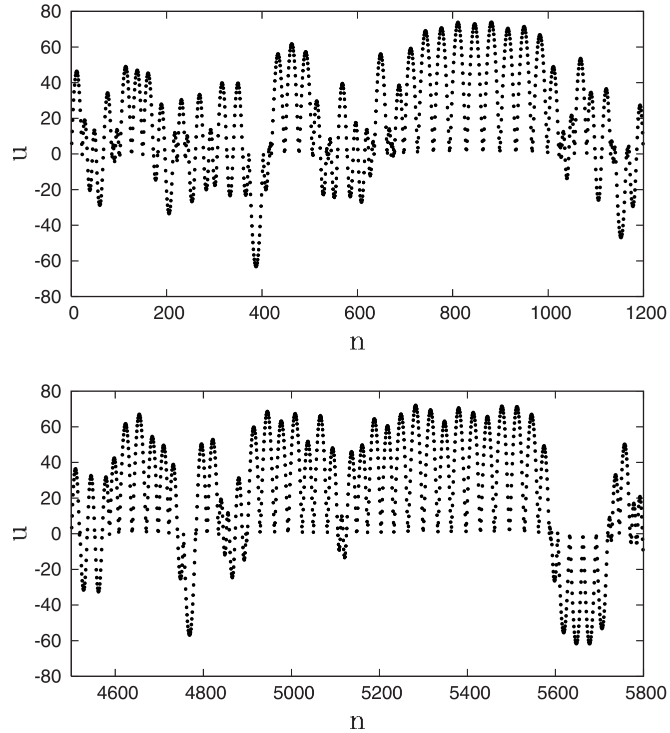

FIG. 3. Time series of the $\{u_n\}$ obtained numerically from g^n. The parameter Q is the same as in Fig. 2.

$\alpha=2v_{\text{ph}}\omega_B/\omega$, $\alpha=1.0$, and $N=[kL], N=2\times10^3$. The parameter b is typically of order $10^{-4}-10^{-2}$. We will set $b=1\times10^{-3}$ below. In the Earth magnetosphere, electrons pass through the wave packet repeatedly with time intervals of $T=L/v_z$. In our case, $L\simeq10^9$ cm, and $v_z\simeq3\times10^{10}$ cm/s, therefore $T\simeq0.033$ s. Now by Eq. (38) we find $Q\simeq2\pi10^{-2}$, and expression (52) gives the characteristic time, $t_d\simeq6$ min for establishing the energy spectrum. One evaluates the characteristic value of particle energy, E. In consideration of Eqs. (34) and (37), from Eq. (42) the following relation results:

$$E_b=\alpha m\left(\frac{1-v_{\text{ph}}}{3}\cdot N^2 b\right)^{2/5}, \tag{54}$$

which well determines the upper value of the energy spectrum.

Substituting in this formula typical values for v_{ph},α, N, and b, we obtain $E_b\simeq8$ MeV.

We conclude that for typical values of the wave field in the Earth's radiation belt, significant diffusion in energy occurs on time scales of the order of a few minutes for electrons with energies up to 8 MeV. This result is in reasonable agreement with the experimental data.[13–15]

IV. PHASE MODIFICATION AND INTERMITTENT DYNAMICS

One studies the dynamic of electrons in an intense wave packet. Shown in Fig. 2 are the phase diagrams, and Fig. 3 shows the evolution of the system in time. These pictures indicate that the system demonstrates both chaotic and regular dynamics. We assume as a heuristic argument that the motion may be described as the composition $G^t\circ g^n$, where g^n is the map (36), and G^t is a certain Hamiltonian flow acting on the smooth manifold M^2. This G^t is to be found. Presumably, the action of G^t is associated with an adiabatic motion. The original Hamiltonian of the system (45) can be rewritten in the form

$$h(\psi,u,\zeta)=Q\cos\psi\sum_n\delta(\zeta-n)+Q(9\pi^2/2Q^2)\times(u/\pi)^{1/3},\quad n\in Z, \tag{55}$$

where the sign of u must be kept fixed. First one counts up the variation of h in each iteration, $\Delta h=\int(dh/d\zeta)d\zeta$,

$$\Delta h=\frac{3}{2}\pi\left(\frac{\pi}{u_n}\right)^{2/3}\sin\psi_n. \tag{56}$$

In deriving Eq. (56), we have used Eqs. (46) along with the further result $dh/d\zeta=\partial h/\partial\zeta+[h,h]=\partial h/\partial\zeta$, and carried out the well-known transformation,[16]

$$\int_{-1}^{1}f(x)\delta'(x)dx=-\int_{-1}^{1}f'(x)\delta(x)dx. \tag{57}$$

It appears to be reasonable that conditions for adiabatic motion,

$$\Delta h/h\ll1, \tag{58}$$

are realized provided that the following relations are valid:

$$\pi^{5/3}/u^{2/3}Q\ll1,\quad Q/u\ll1, \tag{59}$$

and $\Delta h/h=0$ if

$$\sin\psi_n = 0. \tag{60}$$

In that case, the equations of motion (46) reduce to the closed set of ordinary differential equations, describing the autonomous Hamiltonian flow G^t,

$$G^t{:}\dot{u} = [u,h] = Q\sin\psi, \quad \dot{\psi} = [\psi,h] = (3\pi^{5/3}/2Q)u^{-2/3}, \tag{61}$$

associated with the Hamiltonian, $h(\psi,u)$,

$$h(\psi,u) = Q\cos\psi + Q(9\pi^2/2Q^2)(u/\pi)^{1/3}, \quad n\in Z, \tag{62}$$

being the adiabatic invariant of the motion.

Thus, expressions (59) and (60) are the necessary and sufficient conditions for the adiabatic approximation.

Now Eq. (62) describes the orbit, or rather the regular part of a single trajectory, which springs up according to Eq. (60) at $\psi=0$.

As far as the eigenvalues of h are given by that of (ψ,u) on the strange attractor, h will takes its values from the interval only,

$$h \in \{H|h_1 < h < h_0\}, \quad h_1 = Q, \quad h_0 = Q(1+9\pi^2/2Q^2). \tag{63}$$

So far as the eigenvalues of h pertain to the interval, it appears from Eq. (61) that Eq. (62) along with Eq. (63) determine the family of concurrent curves, which corresponds to the continuum of regular orbits. Thus, the dynamics is realized on a certain invariant set, embedded in the phase space, which is to be the connected sum consisting of the strange attractor and continuum of orbits.

Each element of h is assigned to a certain proper orbit with the probabilistic measure d_h. As a consequence, the density of states in the space of orbits may be given as

$$\rho(h) = \tau_h = d\Gamma(h)/dh, \tag{64}$$

the τ_h is the characteristic time of motion along the orbit,[11] and $\Gamma(h)$ is the phase area bounded by the invariant curve, C_h,

$$\Gamma(h) = \oint_{C_h} u(h)d\psi. \tag{65}$$

Then, invoking Eq. (62), we obtained

$$\rho(h) = \tau_h = (6\pi^2/Q)(2Q^2/9\pi^2)^3(h^2/Q^2)(1+Q^2/2h^2). \tag{66}$$

It follows that the orbit associated with $h=h_0$ corresponds to the modal state in orbit space. The regular orbits are well-distinguished, however, by virtue of the fact that Eqs. (56) and (66) transitions from one to another occur at random.

To describe this process, it is pertinent to introduce the new variables—the action J and angle ϑ—which are immediately related with the phase area through the Poincare integral invariant,

$$\oint_{C_h} u(h)d\psi = \oint_{C_H} J d\vartheta. \tag{67}$$

Now we write down

$$\begin{aligned} &J = \Gamma(h)/2\pi, \\ &\omega_h = 2\pi/\tau_h = (Q/3\pi)\,(9\pi^2/2Q^2)^3)(Q^2/h^2 + Q^2/2), \end{aligned} \tag{68}$$

where $\Gamma(h)$ ad τ_h are given by Eqs. (65) and (66), respectively. In the new variables, a Hamiltonian of the problem is found to be

$$H(J) = J\omega_h = \frac{1}{3}h\cdot\frac{1+3Q^2/2h^2}{1+Q^2/2h^2} \tag{69}$$

and the equations of motion are

$$\dot{J} = [J,H] = 0, \quad \dot{\vartheta} = [\vartheta,H] = \omega_h. \tag{70}$$

These equations imply that the motion on a selected level of h is completely determined by own frequency ω_h, and J is to be the invariant of motion. In accordance with Eqs. (63) and (69), the values of J run from π to $J_0=J(h=h_0)$ as $t\to\infty$, and transition from one orbit to another is due to the nonadiabatic behavior of orbits near the strange attractor. To study the drift in the orbit's space, we first define $\Delta J=(\delta J/\delta H)\Delta H = \Delta H/\omega_h$, which is the change in J on each time step. Since H is a function of h, the expression $(\Delta J/J)^2 \simeq (\Delta h/h)^2$ determines the relative rate of filling the phase space. Again, because of Eq. (58) we consider the relation

$$\varepsilon = (\Delta J/J)^2 (\ll 1) \tag{71}$$

as the small parameter of the problem. The same holds for the ratio $\tau_h/\tau_d, \tau_d \simeq \tau_h\varepsilon^{-1}$, where τ_d is the characteristic time of diffusion in J. As a result, we can put into operation the phase average as

$$\langle . \rangle = 1/2\pi\int_{-\pi}^{\pi} d\vartheta. \tag{72}$$

Now the evolution into the orbit's space may be described as a diffusion process by a probability density, $\rho(\vartheta,J,t)$, which obeys the Liouville equations

$$\frac{\partial\rho}{\partial t} + [\rho,H] = 0. \tag{73}$$

By virtue of Eq. (71), we seek the solution to Eq. (73) using one of the methods of perturbation theory.[11] The following one will be employed in this paper. In view of Eq. (71), ρ allows for the following representation:[17]

$$\begin{aligned} \rho(\vartheta,J,t) = \rho(\vartheta,J,t{:}J=\text{const}) + \frac{\partial\rho}{\partial t}\tau_h + \dot{\vartheta}\frac{\partial\rho}{\partial\vartheta}\tau_h + \frac{\Delta J}{\tau_h}\frac{\partial\rho}{\partial J}\tau_h \\ + \frac{1}{2}\left(\frac{\Delta J}{\tau_h}\right)^2\frac{\partial^2\rho}{\partial J^2}\tau_h^2. \end{aligned} \tag{74}$$

Subject to Eqs. (71) and (74), and keeping with Eq. (60), we write down Eq. (73) in the first approximation as

$$\frac{\partial\rho}{\partial t} + \omega_h\frac{\partial\rho}{\partial\vartheta} = 0. \tag{75}$$

The next approximation gives the FPK equation governing an irreversible process in the space, namely

$$\frac{\partial\rho(J,t)}{\partial t}=\frac{1}{2}D(J)\frac{\partial^2\rho}{\partial J^2},\quad D(J)=\frac{\langle(\Delta J)^2\rangle}{\tau_h},\tag{76}$$

where $\rho(J,t)=\langle\rho(\vartheta,J,t)\rangle$, and $D(J)$ is the coefficient of diffusion in J. In deriving Eq. (76), we take into account $\langle\Delta J\rangle=0$. From Eq. (76) it follows that the uniform distribution over $J, J\in\{\pi,J_0\}$ is established in the time $t_d=2J_0^2/D(J)$. In case $h_0^2/Q^2\gg 1$, but $Q\geqslant Q_c$, this formula yields

$$\tau_d\simeq 16\pi^2/3Q.\tag{77}$$

The value of Q_c will be estimated later. In fact, as we shall see, Q_c corresponds to the value of Q at which the vector field of the system is modified. According to Eqs. (68) and (69), this system demonstrates nonlinear dynamics with frequencies $\omega\simeq J^{-2/3}$. So far these frequencies are incommensurable, therefore the set of orbits is dense almost everywhere, and by virtue of Eq. (66) the system is ergodic, and the entropy gain is given by $\Delta S=\ln\Gamma(h)$.[12,18] In our case, $\Delta S=\ln(9\pi^2/2Q^2)$ near the phase transition. Now we discuss spectral properties of the system. The solution for Eq. (75) may be given as

$$\rho(t)=c_h e^{i\omega_h t}e^{-i\vartheta},\tag{78}$$

where ω_h is the proper frequency at a given value of h, and c_h denotes the weighting coefficient. Then we carry out the Fourier transformation, $\rho(\omega)=\int\rho(t)e^{-i\omega t}dt/2\pi$, to find the power spectrum, $\rho(\omega)$,

$$|\rho_\omega|^2=2\pi c_h^2\delta(\omega-\omega_h).\tag{79}$$

Integrating Eq. (79) over the spectrum, we get

$$\int|\rho_\omega|^2 d\omega=2\pi c_h^2(\omega_h).\tag{80}$$

Here, $h\in\{H\}$, where $\{H\}$ is given by Eq. (63), ω_h is the spectrum of proper frequencies, and consequently, $c_h(\omega_h)$ is the spectral resolution. Finally, the spectral resolution $c_h(\omega_h)$ is to be found. Recalling that ω_h is a one-valued function of h for calculating $c_h(\omega_h)$, we apply the one-to-one correspondence

$$c_h(\omega_h)=\rho(h)dh/d\omega_h.\tag{81}$$

Then from Eqs. (66) and (81), the following equation results:

$$c_h(\omega_h)=\mathrm{const}\,\omega_h^{-5/2},\tag{82}$$

which describes the so-called low-frequency $1/\omega$ noise with the divergence proportional to $\omega^{-5/2}$. Note that in our case the spectrum is bounded below by the marginal frequency, ω_0,

$$\omega_0=3\pi/2Q.\tag{83}$$

As known, the $1/\omega$ spectrum is supposed to be a generic property of any dynamic systems demonstrating an intermittent behavior.[19]

We discussed evolution of the system in the orbit's space. Now we consider the problem in the u-state space. We expect that the rate of this process is comparable to the rate of orbital drift. We will show it directly. First, we calculate via Eq. (56) the u changes, $\Delta u/\pi$, in each iteration, to find

$$\Delta u/\pi=(Q/\pi)\sin\psi.\tag{84}$$

By virtue of the second inequality in Eq. (59), these changes are much smaller than characteristic values of u/π. Then, the rate of diffusion in u may be calculated as before,

$$D(u)=\left\langle\left(\frac{\Delta u}{\pi}\right)^2\Big/\tau_h\right\rangle=\left(\frac{Q^2}{2\pi^2\tau_h}\right).\tag{85}$$

Similarly, the characteristic time for establishing the u spectrum is

$$\tau_d=2u_0^2/D(u)\pi^2=4u_0^2/Q^2\tau_h.\tag{86}$$

Here, u_0 is the highest value of u,

$$u_0=\pi(1+2(2Q^2/9\pi^2))^3,\tag{87}$$

which evaluated at $h=h_0, h_0$ is given by Eq. (63). From Eqs. (86) and (87) it follows that $\tau_d\simeq 16\pi^2/3Q$ for the small value of Q. This result agrees with that given by Eq. (77). We will estimate these quantities by setting in Eqs. (86) and (87) $Q=3\pi/\sqrt{2}$ (this corresponds to the extreme value of h_0) to obtain one by one

$$\tau_d=2\times 10^4,\quad u_0=27\pi,\tag{88}$$

which are in good agreement with the results obtained numerically.

It is worth noting that the behavior of the system resembles the so-called intermittency phenomenon described in Ref. 20, and in the relativistic case in Ref. 2. However, there exists a substantial qualitative difference between these processes. Thus the evolution of a system demonstrating the intermittent dynamics can be represented as a sequence of randomly connected parts of regular oscillations. In this case, an initially stable periodic motion loses its stability at a certain value of a control parameter, while in our case the phenomenon is caused by chaotic-order transition, and the quasiperiodicity in the regular motion occurs due to the intrinsic features of strange attractor.

In order to study this aspect of the problem, it is pertinent to introduce the new coordinates $(u,\dot{u})$ on an appropriate configuration space, using the invariant of motion (62). In the variables, Eqs. (61) reduce to the following dynamic system:

$$\ddot{u}=-\frac{\partial w(u)}{\partial u},\quad w(u(h))=\frac{81\pi^2}{8Q^2}\left(\frac{u}{\pi}\right)^{2/3}-\frac{9h}{2Q}\left(\frac{u}{\pi}\right)^{1/3},\tag{89}$$

$$\tfrac{1}{2}(\dot{u})^2+w(u)=W.\tag{90}$$

Here W is the new invariant of motion whose proper values are given by the initial values on SA. It is known[11] that the $(u,\dot{u})$ space may be supplied by the Riemann metrics, $\sqrt{W-w(u)}du$, therefore that is a configuration space as required. It allows for an interpretation of the $w(u)$ as a function describing a certain scalar field with the mean other than zero. Thus Eqs. (89) and (90) are formally identical to equations of motion of a particle having the mass equal to 1 in the potential field $w(u)$. So far as $w(u)$ is a function of h, whose

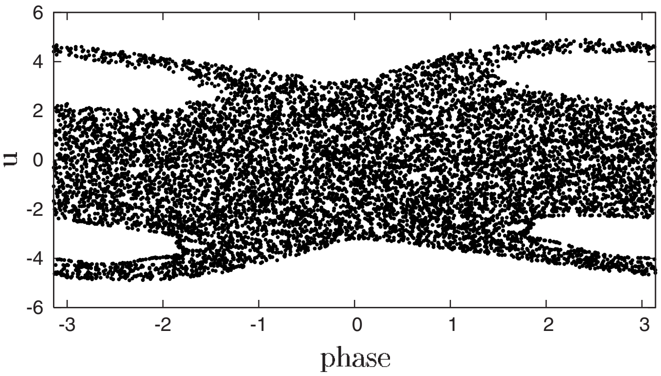

FIG. 4. Numerical solution of the map showing the appearance of an elliptic point. The initial conditions are the same as in Fig. 1, $Q=3/4$. The diagram illustrates the phase transition as the control parameter Q is increased.

proper values belong to SA, consequently the attractive property $w(u)$ is actually due to the SA.

Figure 4 indicates that the phase modification must be associated with the appearance of fixed elliptic points, when an initial saddle point of the attractor changes to a stable elliptic point as the control parameter Q increases above the appropriate critical value, Q_c. Indeed it follows from Eq. (61) that these points are governed by the expressions

$$\sin\psi_e=0, \quad 3\pi^{5/3}/2Qu_e^{2/3}=2\pi, \quad u_e=\pi. \tag{91}$$

These equations are satisfied provided that Q takes the value

$$Q=Q_c, \quad Q_c=3/4. \tag{92}$$

The test does predict the phase modification rather well. This indicates that the sudden appearance of the regular orbits is due to the bifurcation of the vector field. As the wave amplitude is increased to a value slightly above a critical value Q_c, corresponding orbits circulate around the elliptic point or cover the full range of phase angles while remaining within the interval, $u\in(\pi,\pi+4\pi Q^2/3\pi^2)$. The latter follows from Eq. (87) for $Q<3\pi/\sqrt{2}$, but $Q\geqslant Q_c$. It avails oneself the result (38) to evaluate the critical value of the wave field, b_c. In view of Eq. (92), we get

$$b_c=(3/4\pi)^{5/2}(1-v_{\mathrm{ph}}/3)^{3/2}\cdot N^{1/2}, \tag{93}$$

and as appears from the equation, $b_c\simeq 3.5\times10^{-2}$ for $v_{\mathrm{ph}}=0.25$, $N=10^2$. Then substituting in Eq. (54) the value b_c computed just now yields energy $E_b\simeq 3$ MeV. Now we make use of Eq. (77) to find the characteristic time for establishing the energy spectrum, $\tau_d\simeq 70\tau_h$, where τ_h is measured in the time steps, T. Keeping in mind that $N=2\pi k/\Delta k\simeq 2\pi\omega/\Delta\omega$ at $v_{gr}\simeq v_{\mathrm{ph}}$, we get $N=\omega T$. Assuming $\omega=10^4$ s^{-1}, and one takes into account the values of N and v_{ph} used just now, we find $T\simeq10^{-2}$ s, consequently $\tau_d\simeq 0.7$ s.

Calculating the mean heating rate, we obtain $E_b/\tau_d\geqslant 4$ MeV/s, which is much larger than that in the regime mentioned above. This mechanism may be responsible for the prompt energization of relativistic electrons interacting with a narrowband whistler wave packet.[21,22] This could also explain a short time scale of flux enhancement of relativistic electrons reported by Nagai and co-workers.[23]

V. SUMMARY

We have derived the group of transformations on a manifold, which describes electron motion in a coherent whistler wave packet. All solution trajectories belong to the strange attractor (SA), consequently the motion is chaotic, the means (the observables) of which are stable, and their values, irrespective of an initial condition, are independent of time. Because the canonical status of the variables on SA is established, the use of a Poisson bracket leads immediately to the Fokker–Planck–Kolmogorov equation, which describes the evolution of the system to a certain steady state. The rate of the irreversible process predetermines the heating rate of electrons, and the structural invariant on SA yields the upper value of the energy spectrum. In an application of our results to the Earth's radiation belts, this implies that a stable energy distribution in the collisionless plasma of the belts is established on time scales of the order of a few minutes for electrons with energies up to 8 MeV.

The particle motion is no longer completely random as a control parameter of the problem exceeds a certain critical value. In this case, the system demonstrates the intermittent dynamics caused by bifurcation of the vector field. Now, the motion is described by the composition of the map g^n and the Hamiltonian flow G^t, which conserves a defined adiabatic invariants. The sufficient and necessary conditions for that region of phase space where the motion could be approximated by regular orbits have been derived. The SA ensures the stability of a system and causes nonadiabatic behavior of the orbits, which results in orbital drift. As a consequence of that, an invariant set of solutions was found to be connected to some consisting of the SA and continuum of orbits. This feature of motion appeared as the $1/\omega$ noise in the wave spectrum. Our results indicate that this dynamic regime can be realized in the Earth's radiation belt provided that the magnitude wave field, b, is large enough $(b>10^{-3})$.

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